Analytical Dynamics and Contact Analysis

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Abstract

Robots in a vacuum are useless— we care about robots insomuch as they can be used to interact with the environment. Understanding contact forces, their analysis within a robotic system, and how they can be controlled is, therefore, critical. This document discusses why we need the Analytical Dynamics of Lagrange, Hamilton, and Gauss for robot analysis and builds to a discussion of contact forces in a dynamical system. We'll see that Lagrangian mechanics provides a coordinate independent approach to classical mechanics, liberating Newton's second law $\boldsymbol{f} = m\boldsymbol{a}$ from its inertial reference frames and making it applicable to the generalized coordinates of robotic joint angles. We'll also see how both the conservation of momentum and the conservation of energy are fundamental consequences of the equations themselves, and how contact constraints in the environment can be modeled using constrained optimization techniques. This discussion culminates in an analysis of contact constraints for floating-based rigid body robotic systems and derive the constrained equations of motion that form the backbone for controller used in today's most advanced robotics systems.

1 Why analytical dynamics?

A staple in classical mechanics is Newton's famous equation $\boldsymbol{f} = m\ddot{\boldsymbol{x}}$ relating a force $\boldsymbol{f}$ exerted on a point particle to its resulting Cartesian acceleration $\ddot{\boldsymbol{x}}$. Technically, that, in addition to Newton's third law that says forces come in equal and opposite pairs, is all we need to know to analyze the behavior of classical systems.[1] Why would we need anything more sophisticated?

1.1 Computing with Newton's laws

Newton's laws only accurately describe motion in an inertial reference frame.[2] When we describe the motion of a collection of 3-dimensional particles $\boldsymbol{x}_i \in \mathbb{R}^3$,

[1] Newton's first law is effectively just a special case of the second law Susskind (2011). With no force ($\boldsymbol{f} = 0$), the system doesn't accelerate.

[2] An inertial reference frame is one that isn't itself accelerating. Technically such a frame never exists since gravity, for instance, never stops exerting its influence, and electrostatic

this restriction doesn't cause problems. Each of the particles lives in the 3-dimensional Cartesian space and we can choose some non-accelerating Cartesian frame as our inertial reference to describe the the system's behavior.

But generally we run into two problems when using this analysis. First, there are on the order of 10^{23} particles in everyday macro objects (Avogadro's number), and that's far too many particles to analyze by hand. Even the most advanced supercomputer wouldn't be able to handle such a computation at the speeds we need for control. Second, these particles are highly constrained in their motion. They need to fit into lattice structures to form the shapes of the individual robot parts! Computing specifically how each of these particles under their lattice constraints produce equal and opposite forces to generate a cumulative net behavior of the overall system is intractable and subject to severe numerical precision problems.

Fortunately, since today's robots are constructed from metals with strong lattices, analysts have successfully approximated them in practice as precise rigid structures, which significantly simplifies their behavior. A mathematically precise rigid body is fully characterized by the behavior of its center of mass. If we know the position, velocity, and acceleration of the center of mass, as well as the rotational counterparts to those quantities at the center of mass, we know everything there is to know about the behavior of that rigid body system. Thus, we can reduce the 10^{23}-ish constraints to just a tiny handful, on the order of the number of joints in the robot, that describe simply how each rigid body attaches to other rigid bodies in the system at joints.

Now we can computationally analyze the equal and opposite interactions of forces at these joints using Newton's analysis to derive fast algorithms to recursively compute how each of the rigid body parts of the robot must be accelerating given that we've applied specific torques at the joints and vice versa.[3]

1.2 The equations of motion and generalized coordinates

But although such a Newton-based algorithm calculates the forward dynamics $\ddot{\boldsymbol{q}} = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\tau})$, where $\boldsymbol{\tau}$ is the collection of torques exerted on the joints, and even the corresponding inverse dynamics $\boldsymbol{\tau} = f^{-1}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, it doesn't give us the function f, itself. The algorithm evaluate the function both forward and backward, but it doesn't give us anything we can analyze. In order to derive control algorithms, we need to understand mathematically how forces relate to accelerations in order to calculate things like how an objective function over $\ddot{\boldsymbol{q}}$ transforms into a function over $\boldsymbol{\tau}$. Control algorithms require explicit representations of the equations of motion f.

forces are also prevalent. It's a good approximation, though, at a macroscopic level, especially when explicitly modeling the influence of a gravitational pull.

[3]This algorithm, knows as the Recursive Newton-Euler (RNE) algorithm, is the fastest algorithm we know for computing the dynamics of a rigid-body robot Siciliano et al. (2010), and it, in conjunction with Featherstone's convenient spacial coordinates for representing positional and rotational components in a single 6-dimensional vector, form the basis of all state-of-the-art dynamic robotic systems Featherstone (2008).

Unfortunately, we are fundamentally bared from directly using Newton's three laws of motion to derive the equations of motion. Ideally, we'd want to represent the equations in terms of the robot's joint angles (or joint positions) $\boldsymbol{q} \in \mathcal{C} \subset \mathbb{R}^d$ since these numbers minimally represent the configuration of the system. Such a minimality of representation means that we can consider $\boldsymbol{q}$ to be a set of *generalized coordinates* for the system. Despite the huge number of particles in the robotic system (on the order of 10^{23}), there are so many constraints on the system due to the robot's rigid body structure, all realizable configurations (up to our rigid body approximation) can be represented by the d coordinates $\boldsymbol{q}$. And the number of joints d is typically on the order of tens (e.g. around 7 for rooted manipulators or around 25 to 30 for floating based balancing humanoids).

These coordinates $\boldsymbol{q}$ are far from an inertial reference frame! In order to move the end-effector in an straight line along the Cartesian velocity vector $\dot{\boldsymbol{x}}$, we need to move the joints in a sophisticated pattern given by the pseudoinverse of the forward kinematic map's Jacobian. Mathematically, if $\phi : \mathcal{C} \to \mathbb{R}^3$ denotes the forward map from the joints to the end-effector, then since $\dot{\boldsymbol{x}} = \boldsymbol{J}_\phi \dot{\boldsymbol{q}}$ we need to instantaneous follow $\dot{\boldsymbol{q}} = \boldsymbol{J}_\phi^\dagger \dot{\boldsymbol{x}}$. Integrating this process forward makes the robot's end-effector move in a straight line, but the trajectory traced out in $\boldsymbol{q}$ is far from a straight.

More than that, if we find a joint space trajectory that moves the end-effector in a straight line, other points on the robot's body will still be moving in curves through the Cartesian space. The constraints on the robot's structure enforce that we will always have this problem. There's no way in general to get all points on the robot's body to simultaneously move in a straight line using movements of the joints. The map ϕ is fundamentally nonlinear, and that nonlinearity imposes an inherent curvature to the manifold of configurations. We won't discuss these geometric ideas in depth, but they reveal that the space of joint angles cannot be an inertial reference frame. And that means Newton's laws do hold in their basic form. Leveraging Newton would require somehow nonlinearly transforming the laws into this lower-dimensional warped space from an inertial frame within the Cartesian space of point particles on the robot's body.

Fortunately, Lagrangian mechanics is precisely the generalization we need to perform such a calculation with relative ease.

2 Lagrangian mechanics and invariance to coordinate transform

Newton's second law on a system of particles $\{\boldsymbol{x}_i\}_{i=1}^n$ moving through Cartesian space $\boldsymbol{x}_i \in \mathbb{R}^3$ can be written collectively in matrix form as

$$\underbrace{\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_n \end{bmatrix}}_{\boldsymbol{f}\in\mathbb{R}^{3n}} = \underbrace{\begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}}_{\boldsymbol{M}\in\mathbb{R}^{(3n)^2}} \underbrace{\begin{bmatrix} \ddot{\boldsymbol{x}}_1 \\ \ddot{\boldsymbol{x}}_2 \\ \vdots \\ \ddot{\boldsymbol{x}}_n \end{bmatrix}}_{\ddot{\boldsymbol{x}}\in\mathbb{R}^{3n}}, \tag{1}$$

or, more compactly, as $\boldsymbol{f} = \boldsymbol{M}\ddot{\boldsymbol{x}}$ as indicated.

Another way to describe that equation is to define a variable $\boldsymbol{p} = \boldsymbol{M}\dot{\boldsymbol{x}}$ representing the *momentum* of the system. Then this simple equation says that a force $\boldsymbol{f}$ gives (or creates) the time-rate of change in the momentum $\boldsymbol{p}$ of the system, or $\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt}$.

Additionally, if the force is conservative,[4] we can say that the force is the negative gradient of some position based potential $\mathcal{V}(\boldsymbol{x})$:

$$\boldsymbol{f} = -\nabla\mathcal{V}(\boldsymbol{x}). \tag{2}$$

For instance, if $\mathcal{V}$ the gravitational potential energy[5] $\mathcal{V}(\boldsymbol{x}) = \sum_i m_i \boldsymbol{g}^T \boldsymbol{x}_i$, then the negative gradient of this potential is

$$-\nabla\mathcal{V}(\boldsymbol{x}) = \begin{pmatrix} -m_1\boldsymbol{g} \\ -m_2\boldsymbol{g} \\ \cdots \\ -m_n\boldsymbol{g} \end{pmatrix}. \tag{3}$$

This negative gradient is a vector of gravitational forces, one for each of the n particles in the system. Generally, $\mathcal{V}(\boldsymbol{x})$ can be any differentiable potential function of position $\boldsymbol{x}$.

Thus, we can rewrite the second law of motion in Cartesian space as

$$\boldsymbol{f} = \boldsymbol{M}\ddot{\boldsymbol{x}} \quad \rightarrow \quad -\nabla\mathcal{V}(\boldsymbol{x}) = \frac{d\boldsymbol{p}}{dt}. \tag{4}$$

[4] Non-conservative forces, such as friction, are actually the result of interactions with unmodeled parts of a larger external system. Friction, for instance, is the electromagnetic interaction with microscopic particles on a surface. In this theoretical setting, the assumption that the force is conservative is thereby tantamount to assuming that we model the full mechanical system.

[5] Typically, in undergraduate courses we see the potentially energy written as mgh, where m is the mass and both g and h are scalar values representing the gravitational constant and the height of our particle, respectively. In this case, if $\boldsymbol{g} = [0, 0, g]^T \in \mathbb{R}^3$ is a vector pointing upward (the potential increases upward), then $\boldsymbol{g}^T\boldsymbol{x}$ is the component of $\boldsymbol{x}$ in the upward direction (i.e. the height) scaled by the norm of $\boldsymbol{g}$. Specifically, it is $x_z g$ when $\boldsymbol{x} = [x_x, x_y, x_z]^T$, so the equation we give here is a generalized form of the familiar potential energy.

Noting also that the momentum itself, is the velocity gradient of the kinetic energy $\mathcal{T}(\dot{\boldsymbol{x}}) = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{M}\dot{\boldsymbol{x}}$, we can rewrite the equation as

$$-\nabla_{\boldsymbol{x}}\mathcal{V}(\boldsymbol{x}) = \frac{d}{dt}\nabla_{\dot{\boldsymbol{x}}}\mathcal{T}(\dot{\boldsymbol{x}}). \tag{5}$$

To be more general, lets rewrite this equation in terms of a function $\mathcal{L}$ combining both the kinetic and potential energies

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \mathcal{T}(\dot{\boldsymbol{x}}) - \mathcal{V}(\boldsymbol{x}). \tag{6}$$

This function, known as the *Lagrangian*, compactly encodes all of the properties of the system as we'll explore below.

Equation 5, rewritten in terms of the Lagrangian, is

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}} = 0. \tag{7}$$

This equation is called the Euler-Lagrange Equation, and when the Lagrangian has the specific form in Equation 6, these partial derivatives are $\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{V}}{\partial \boldsymbol{x}}$ and $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}} = \frac{\partial \mathcal{T}}{\partial \dot{\boldsymbol{x}}} = \boldsymbol{p}$, and this more general variant reduces exactly to Equation 5.

We write Newton's equations in terms of the Lagrangian because although we've derived this equation as the difference between kinetic and potential energy for this mechanical setting, there are other systems (e.g. electromagnetic) for which the Lagrangian takes on other forms that may not have this interpretation. Some systems may even have Lagrangians that vary with time. Many of the theoretical properties that we drive below, therefore, hold beyond the relatively simple setting of mechanical systems of concern in robotics; the Lagrangian function is of central importance to many branches of modern physics.

This reformulated equation of motion given in Equation 7 both generalizes Newton's equation and, as we'll see in the next section, also establishes an explicit connection to a particular *functional* optimization problem. All of the magic of Lagrangian mechanics and its invariance to coordinate transformations derives from the observation that this Euler-Lagrange equation is a first-order optimality condition for *functional* optimization as analyzed by the Calculus of Variations.

2.1 The principle of stationary action

It's beyond the scope of this document to fully discuss the Calculus of Variations and (see Gelfand & Fomin (1963); Susskind (2011); Taylor (2005)), but the following subsection should provide some high-level context, especially by analogy to the more familiar finite-dimensional optimization.

2.1.1 First-order optimality for functional optimization

Let $\psi : \mathbb{R}^k \to \mathbb{R}$ be some k-dimensional objective function. The first-order optimality condition for this objective is the statement that the gradient of the

function must be zero for us to be at one of the local optima: $\nabla\psi = 0$. This critical point may not be a minimum (it may be a maximum or a saddle point, which is some mixture of minima and maxima in different dimensions), but we know that the gradient must be zero at a minimizer.

Suppose now that instead of a function in k dimensions we have a function of uncountably infinite dimensions. What does this mean? In order to index a dimension of our input vector, we need to use a real number (the canonical representation of an uncountably infinite set). Denoting that index by $t \in [0, T]$, we can represent a vector in our infinite-dimensional space by a function mapping this index range to some finite-dimensional space[6] $\boldsymbol{x} : [0, T] \to \mathbb{R}^d$. In other words, each input to this infinite-dimensional function is a trajectory ranging in time from 0 to T.

Such functions that take other functions as input and output a real value are referred to as *functionals*. Giving them their own name reminds us that we need to be extra careful when discussing some of the properties since there are so many degrees of freedom; limiting arguments, for instance, become critical and introduce some edge-case issues. Despite that, the basic approaches to optimization, especially analytical optimization, strongly parallel finite-dimensional optimization. In particular, there's an analogous first-order optimality condition which again effectively says that the gradient of the functional must be 0 at a critical point. In this case, though, we need an appropriate generalization of "gradient" for infinite-dimensional function spaces.

In analogy to the finite-dimensional case, we expect the functional gradient to be another time-indexed function (continuously infinite-dimensional vector) of the form $g : [0, T] \to \mathbb{R}$. Making the first-order optimality condition $g(t) = 0$ for all t. It's beyond the scope of this document to give a full derivation of the functional gradient, so we'll just state and discuss the result. There are numerous very good introductions to the Calculus of Variations presented both in a general mathematical context and specifically in the context of physics (see Gelfand & Fomin (1963); Susskind (2011)).

Suppose the objective functional took the form

$$\mathcal{F}[\boldsymbol{x}] = \int_0^T f(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt. \tag{8}$$

Explicitly, this functional is an integral of individual costs at each time t in the time range. This form of functional describes numerous practical problems (for instance, minimum distance or minimum tension problems), but we particularly care about it here because of how its first-order optimality condition relates to Equation 7.

For the class of objective functionals given by Equation 8, the first-order

[6] Giving the range space of this function more than one-dimension is analogous to allowing finite-dimensional functions to be defined over spaces of matrices rather than strictly over column vectors, where each row index (analogous to the index t) maps to a vector of d numbers rather than just a single real value.

optimality condition is

$$\underbrace{\frac{\partial f}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial f}{\partial \dot{\boldsymbol{x}}}}_{\text{Func. gradient}} = 0 \qquad \text{For all time } t. \tag{9}$$

the left-hand side of the equation, as indicated, is the functional gradient of $\mathcal{F}$ evaluated at a particular time index t.

Now, returning to the above Euler-Lagrange equation for dynamical systems in Equation 7, we see that that equation, describing the time evolution of the system, is exactly the first-order optimality equation for a functional of the form given in Equation 8 with f being the Lagrangian $\mathcal{L}$. This connection forms the basis of Lagrangian mechanics and can be stated as the following principle: **The physically realized dynamical behavior of a Newtonian system developing over time is a stationary point of the *action* functional**

$$\mathcal{A}[\boldsymbol{x}] = \int_0^T \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt. \tag{10}$$

This principle is called the Principle of Stationary Action.[7] This principle comes simply from noticing that the Euler-Lagrange equation is the functional first-order optimality condition given by Equation 9 of the action.[8]

2.1.2 Coordinate invariance

This principle of stationary action is a physical principle depending only on the physical time-varying sequence of points traced out by the trajectory, and not on the specific coordinate system within which we choose to represent these points. Physics as described in Cartesian coordinates, for instance, should be the same as the physics described in polar coordinates, and both of those descriptions should ultimately match the (slightly stranger) description of the physics from the perspective of someone sitting on a merry-go-round.

More abstractly, suppose we've decided to change from some set of coordinates $\boldsymbol{x} \in \mathbb{R}^d$ to a new set $\boldsymbol{q} \in \mathbb{R}^d$. And suppose we have some invertible (bijective) coordinate transformation function $\phi(\boldsymbol{q}) = \boldsymbol{x}$ that takes us from

[7] Historically, this principle has been called the Principle of Least-Action based on its connection to optimization. But generally, first-order optimality conditions aren't enough to definite a minimum (the function must be positive definite there, too, which only the second-order optimality conditions tell us), and in this specific case, the dynamic trajectories described by this principle aren't actually local minima. They're at indefinite points which are neither minima nor maxima Susskind (2011). However, since the first-order optimality principle states that the functional gradient of this action functional is zero at the realized physical trajectory, we know that no infinitesimal perturbation of the trajectory will change the objective functional at all to the first order.

[8] The general first-order optimality condition of Equation 9, too, is known as the the Euler-Lagrange equation within the Calculus of Variations in connection to what we describe presently. In this document, though, we'll typically reserve the term "Euler-Lagrange equation" to refer specifically to the application of this first-order optimality principle to the action functional of Equation 10.

$\boldsymbol{q}$ back into the coordinate system that we've been using $\boldsymbol{x}$. Then the gradient of finite-dimensional functions $f(\boldsymbol{x})$ transform by the Jacobian of ϕ as $\nabla_{\boldsymbol{q}} f(\phi(\boldsymbol{q})) = \boldsymbol{J}_\phi^T \nabla_{\boldsymbol{x}} f$. Since ϕ is invertible, so is its Jacobian, which means $\nabla_{\boldsymbol{q}} f(\phi(\boldsymbol{q}))$ is zero if and only if $\nabla_{\boldsymbol{x}} f(\boldsymbol{x})$ is zero. In other words, the critical points don't change under such a change of coordinates. The theorem below shows us that functional gradients transform in an analogous manner, allowing us to draw similar conclusions which, in this case, will imply that the Euler-Lagrange equation is a valid description of the system no matter what coordinate system we use.

Although we're currently discussing invertible coordinate transformations, the following lemma derives the mathematical transformation of the Euler-Lagrange equation here more generally in terms of any differentiable map transforming the coordinates. We'll use the more general result later on.

Lemma 1. Transformation of the Euler-Lagrange equation. *Suppose $\phi : \mathbb{R}^d \to \mathbb{R}^n$ is some differentiable map that takes d-dimensional coordinates $\boldsymbol{q}$ to n-dimensional coordinates $\boldsymbol{x}$. Denoting the transformed Lagrangian as $\widetilde{\mathcal{L}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{L}(\phi(\boldsymbol{q}), \frac{d}{dt}\phi(\boldsymbol{q}))$ in terms of the original Lagrangian $\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}})$, the Euler-Lagrange equation transforms as*

$$\frac{\partial \widetilde{\mathcal{L}}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}} = \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\right)\frac{\partial \phi}{\partial \boldsymbol{q}} = 0,$$

where $\frac{\partial \phi}{\partial \boldsymbol{q}}$ is the Jacobian of ϕ.

Proof. (sketch) We first address each of these terms individually, and then later combine them to derive the result. The majority of this proof sketch is just a calculation. Since $\frac{d}{dt}\phi(\boldsymbol{q}) = \frac{\partial \phi}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}$, we have

$$\frac{\partial \widetilde{\mathcal{L}}}{\partial \boldsymbol{q}} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}}\frac{\partial \phi}{\partial \boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\underbrace{\left(\frac{\partial^2 \phi}{\partial \boldsymbol{q}^2}\right)}_{\text{Tensor}}\dot{\boldsymbol{q}}. \tag{11}$$

As indicated in the expression, that last term includes a third-order tensor representing the second derivatives of the map ϕ. It consists of of one Jacobian-sized matrix $\boldsymbol{D}_i$ for each dimension i of $\boldsymbol{q}$ defining how the Jacobian changes when $\boldsymbol{q}_i$ is infinitesimally perturbed. Intuitively, if we scale each $\boldsymbol{D}_i$ by how much we actually change $\boldsymbol{q}$ in that dimension ($\delta \boldsymbol{q}_i$) and then add up all those contributions, the resulting matrix $\sum_i \boldsymbol{D}_i \delta \boldsymbol{q}_i$ will have the same dimensions as the Jacobian and will represent how the Jacobian changes when we move in the direction $\delta \boldsymbol{q}$. In this case, our perturbation $\delta \boldsymbol{q}$ is the trajectory velocity $\dot{\boldsymbol{q}}$, this term represents the time derivative of the Jacobian as we move infinitesimally along the trajectory. We don't do it here, but one can formalize this argument and prove that that rightmost term is exactly

$$\underbrace{\left(\frac{\partial^2 \phi}{\partial \boldsymbol{q}^2}\right)}_{\text{Tensor}}\dot{\boldsymbol{q}} = \frac{d}{dt}\left(\frac{\partial \phi}{\partial \boldsymbol{q}}\right). \tag{12}$$

We'll use this relation below. This relation can also be viewed as an application of the chain-rule to calculating the time derivative of the Jacobian.

Now, continuing on to the second term we can calculate

$$\begin{aligned}\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}} &= \frac{\partial}{\partial \dot{\boldsymbol{q}}} \mathcal{L}\left(\phi(\boldsymbol{q}),\ \frac{\partial \phi}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}\right) \\ &= \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\frac{\partial \phi}{\partial \boldsymbol{q}},\end{aligned}$$

where we've again used the relation $\dot{\boldsymbol{x}} = \frac{d}{dt}\phi(\boldsymbol{q}) = \frac{\partial \phi}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}$.

Therefore, the time derivative is

$$\begin{aligned}\frac{d}{dt}\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}} &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\frac{\partial \phi}{\partial \boldsymbol{q}}\right) \\ &= \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\right)\frac{\partial \phi}{\partial \boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\left(\frac{d}{dt}\frac{\partial \phi}{\partial \boldsymbol{q}}\right) \qquad (13)\end{aligned}$$

Combining these calculations (subtracting the second from the first), we see that the rightmost term of Equation 11 cancels with the Jacobian time derivative term in Equation 13 because of the relationship we discussed above. We get

$$\begin{aligned}\frac{\partial \widetilde{\mathcal{L}}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\boldsymbol{q}}} &= \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}}\frac{\partial \phi}{\partial \boldsymbol{q}} + \overbrace{\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\underbrace{\left(\frac{\partial^2 \phi}{\partial \boldsymbol{q}^2}\right)}_{\text{Tensor}}\dot{\boldsymbol{q}}}^{\text{Cancels}}\right) - \left(\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\right)\frac{\partial \phi}{\partial \boldsymbol{q}} + \overbrace{\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\left(\frac{d}{dt}\frac{\partial \phi}{\partial \boldsymbol{q}}\right)}^{\text{Cancels}}\right) \\ &= \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\right)\frac{\partial \phi}{\partial \boldsymbol{q}}, \qquad (14)\end{aligned}$$

which is the expression we were looking for. □

One immediate consequence of this lemma, is that when ϕ is invertible, since its Jacobian $\frac{\partial \phi}{\partial \boldsymbol{q}}$ is then also invertible, the Euler-Lagrange equation holds in in $\boldsymbol{x}$ if and only if it holds in $\boldsymbol{q}$ as well. Therefore, this generalized reformulation of Classical Mechanics characterizes mechanics in any coordinate system, independent of whether that coordinate system represents an inertial frame of reference.

This means we can choose whatever coordinate system is most convenient for the problem, describe the kinetic and potential energy in that coordinate system to write down the Lagrangian, and derive the equations of motion by simply writing out the Euler-Lagrange equation for that Lagrangian.

The next section shows how this same procedure works for any set of generalized coordinates, even if those coordinates parametrize a highly constrained system as is the case in rigid body dynamics.

2.1.3 Generalized coordinates and system constraints

In the previous section, we saw how framing Newton's laws as an optimality criterion induced a coordinate-frame-agnostic reformulation of the mechanical laws of physics, allowing us to derive equations of motion from the perspective of any coordinate systems, even accelerated frames. However, as presented, all arguments have been specific to simple invertible changes of coordinates. More generally, we want to be able to derive the equations of motion for reduced dimensional generalized coordinate systems that, themselves, implicitly introduce constraints. This section demonstrates that the same Euler-Lagrange equation above holds for the case where the d generalized coordinates parametrize a d-dimensional submanifold of the original Cartesian space, thereby providing a procedure for deriving constrained equations of motion. We start by introducing a general principle of constrained dynamics from an intuitive perspective.

In order to understand the behavior of a system in the constrained case, we need to better understand the notion minimal generalized coordinates. Consider the original fully unconstrained setting discussed above where we have n particles each living in the 3-dimensional Cartesian space. Usually, this $3n$-dimensional Cartesian space of configurations is actually highly constrained. For instance, as described above, the particles might be constrained by the rigid lattice structure of each rigid body part of physical robot, in which case the positions or each of the n particles (where n here is on the order of 10^{23}) may be fully described by a relatively tiny number of robotic joint angles d. In this case, the mapping ϕ from joint angles $\boldsymbol{q} \in \mathcal{C} \subset \mathbb{R}^d$ to the n particle positions $\boldsymbol{x} \in \mathbb{R}^{3n}$ heavily constrains the system.

Imagine choosing a point $\boldsymbol{q} \in \mathcal{C}$ and observing the point $\boldsymbol{x} = \phi(\boldsymbol{q})$ that it maps to. Now consider perturbing that point within $\mathcal{C}$ in an arbitrary direction $\delta\boldsymbol{q}$. The resulting perturbed point can be approximated to the first order using a Taylor expansion

$$\delta\boldsymbol{x} = \phi(\boldsymbol{q} + \delta\boldsymbol{q}) - \phi(\boldsymbol{q}) \approx +\frac{\partial\phi}{\partial\boldsymbol{q}}\delta\boldsymbol{q}. \tag{15}$$

Since the Jacobian $\frac{\partial\phi}{\partial\boldsymbol{q}}$ maps d-dimensional points to $3n$-dimensional points, assuming the mapping is full rank at this point, the space of possible perturbations $\delta\boldsymbol{x}$ is only a d-dimensional subspace of $\mathbb{R}^{3n}$. Intuitively, the map ϕ cannot create volume during a mapping because every point in the image must have a corresponding point in the domain that was mapped there. Therefore, the space of mapped points cannot be higher dimensional than the space that was mapped.

Thus, the collection of points

$$\mathcal{S} = \{\boldsymbol{x} = \phi(\boldsymbol{q}) \mid \boldsymbol{q} \in \mathcal{C}\} \tag{16}$$

is a d-dimensional smooth surface embedded within the much larger $3n$-dimensional space of independent particle positions. In general, the analysis of lower-dimensional generalized coordinate representations of system is equivalent to the analysis of the behavior of a system when constrained to live on that lower-dimensional surface. This gives us a physical coordinate-independent perspective within which

we can discuss how the constrained physics of generalized coordinates play out mathematically.

Imagine a particle $\boldsymbol{x}$ lives on a surface embedded in a high-dimensional Cartesian space.[9] Suppose we view the surface from a fixed inertial frame so Newton's laws hold. If an arbitrary Cartesian force $\boldsymbol{f}$ is applied to the system, by Newton's second law the system must experience an acceleration. However, since the system is constrained to always move along the surface, there must be something enforcing a somewhat deviated acceleration needed to keep the system tracing the surface. The only way the resulting curved trajectory can be consistent with Newton's laws of motion is if the constraint itself imposes this deviation force $\boldsymbol{f}_c$.

In fact, such *constraint forces* appear and can be felt in many everyday situations. If you ever ride a roller coaster (or even just drive a car), each car is constrained to follow the path defined by the track, so as the rider, you feel very strongly each force placed on you by the track (transferred to you through the car) as the roller coaster follows its curved trajectory. Left alone, you would continue in a straight line under your current instantaneous heading (or fall under the force of gravity), but since you're strapped into the car, you feel each of the forces exerted on you to coax you into the ride's loopy route.

What can we say about these constraint forces? Intuitively, the constraint cannot operate tangentially to itself. Otherwise, it could perform work on the system evolving along the surface, which would change its energy. The force induced by the constraint must be entirely orthogonal to the surface. Unfortunately, since we aren't just dealing with two-dimensional surfaces embedded in a three dimensional space, the mathematics of this orthogonality requires slightly more machinery to fully analyze.

Suppose, as above, $\phi : \mathcal{C} \to \mathbb{R}^{3n}$ is our map parametrizing the lower-dimensional surface (every point $\boldsymbol{q} \in \mathcal{C}$ maps to some point $\boldsymbol{x} \in \mathcal{S} \subset \mathbb{R}^{3n}$ on the surface). Then if we currently sit at $\boldsymbol{q}$ (equiv. $\boldsymbol{x} = \phi(\boldsymbol{q})$) on the surface, all perturbations away from the point are given to the first order by

$$\delta \boldsymbol{x} = \frac{\partial \phi}{\partial \boldsymbol{q}} \delta \boldsymbol{q}, \tag{17}$$

where $\frac{\partial \phi}{\partial \boldsymbol{q}} \in \mathbb{R}^{3n \times d}$ is the (very tall and skinny) Jacobian matrix. This equation tells us that we can describe all first-order movements $\delta \boldsymbol{x}$ along the lower dimensional surface as linear combinations of the columns of $\frac{\partial \phi}{\partial \boldsymbol{q}}$. Those columns, which span the Jacobian's column space, define what it means to be "parallel" to the surface at $\boldsymbol{q}$, which in turn defines what it means to be "orthogonal" at that point.

The statement that the constraint force operates only orthogonally to the surface specifically means that it must lie entirely in the null space of ϕ's Jacobian $\frac{\partial \phi}{\partial \boldsymbol{q}}$. Another way of saying that is that the constraint must exert zero

[9] By Cartesian, we mean that it is a produce space of many Cartesian configurations. This space is best described more precisely as a high-dimensional Euclidean space, but we choose to use the term Cartesian to emphasize the constituent spaces from which it was formed.

influence within the *column* space of ϕ. This principle (really its generalization, as discussed below) is known as the d'Alembert's principle.

D'Alembert's principle states that the difference between the observed change in momentum of the constrained system and the applied potential force, from the perspective of the surface's tangent space $\mathbf{span}(\frac{\partial \phi}{\partial \boldsymbol{q}})$ is zero. In other words, any difference

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}}}_{\boldsymbol{f}} - \underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}}_{\dot{\boldsymbol{p}}} = \boldsymbol{f}_c^T, \tag{18}$$

which we can interpret as the force induced by the constraint to keep the system moving consistently with the constraints, must always be orthogonal to the columns of $\frac{\partial \phi}{\partial \boldsymbol{q}}$. In other words, we require that

$$\boldsymbol{f}_c^T \frac{\partial \phi}{\partial \boldsymbol{q}} = \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}}\right)\frac{\partial \phi}{\partial \boldsymbol{q}}}_{\text{Transformed equation}} = 0. \tag{19}$$

However, now compare this expression to the expression in Lemma 1 for how the Euler-Lagrange equation transforms with changes of variable. They're exactly the same. The Euler-Lagrange equation, written in lower-dimensional (constrained) generalized coordinates, implicitly encodes the d'Alembert's principle!

These arguments tell us that to derive the equations of motion of a constrained system, all we need to do is describe the constraint in terms of some minimal generalized coordinates, calculate the kinetic and potential energies in those coordinates to derive the Lagrangian, and then simply write out out the Euler-Lagrange equations taking derivatives with respect to these generalized coordinates. In other words, **the Euler-Lagrange equations just work in any (possibly constrained) generalized coordinate system.**

The above discussion is somewhat more specific to time-independent surface constraints (i.e. fixed holonomic constraints) in order to emphasize intuition, but all of these arguments can be modified to allow the constraint surface move over time by introducing the notion of *virtual displacements*, which are effectively the the space of tangent movements at a fixed moment in time. Rather than diving into that terminology, which can be confusing, we point instead to some of the canonical references discussing the ideas (see Murray et al. (1994); Spong et al. (2005)).

2.2 External forces

All of the above analysis assumed that the system under analysis is a entirely closed system, such that no external forces interact with the system. In robotic systems, though, we need a notion of external force as input into the system in order to design controllers to control the system to a desired state. How do such forces enter into our analysis. We'll see that what we consider "external"

forces are simply forces arising from an unmodeled portion of a larger surrounding system. Here we give a heuristic argument to discuss how external forces manifest mathematically.

Consider a system consisting of two parts, a modeled portion $\mathcal{S}_1$ and an unmodeled portion $\mathcal{S}_2$. Denote the kinetic and potential energies of each portion as $\mathcal{T}_i(\dot{\boldsymbol{q}})$ and $\mathcal{V}_i(\boldsymbol{q})$, respectively for $i = 1, 2$.

This partitioning leads to a combined Lagrangian of the form

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \Big(\mathcal{T}_1(\dot{\boldsymbol{q}}) + \mathcal{T}_2(\dot{\boldsymbol{q}}) \Big) - \Big(\mathcal{V}_1(\boldsymbol{q}) + \mathcal{V}_2(\boldsymbol{q}) \Big) = \mathcal{L}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathcal{L}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}). \quad (20)$$

Writing out the Euler-Lagrange equation, we get

$$\begin{aligned} & \frac{\partial}{\partial \boldsymbol{x}} (\mathcal{L}_1 + \mathcal{L}_2) - \frac{d}{dt} \frac{\partial}{\partial \dot{\boldsymbol{q}}} (\mathcal{L}_1 + \mathcal{L}_2) = 0. \\ \Rightarrow \quad & \frac{\partial \mathcal{L}_1}{\partial \boldsymbol{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}_1}{\partial \dot{\boldsymbol{q}}} = \underbrace{- \left(\frac{\partial \mathcal{L}_2}{\partial \boldsymbol{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}_2}{\partial \dot{\boldsymbol{q}}} \right)}_{\text{External force } \boldsymbol{\tau}} \end{aligned} \quad (21)$$

As indicated in the above equation, since each portion of the Euler-Lagrange equation is in units of force (as transformed into the given generalized coordinate system), we can interpret the right hand side of the equation as an external force. This interpretation allows us to write the Euler-Lagrange equation as

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{\tau}, \quad (22)$$

where $\mathcal{L}$ represents the portion of the system that we're explicitly modeling and $\boldsymbol{\tau}$ is the vector of forces injected into the system from an external unmodeled portion of the larger surrounding system. Notice that in Equation 21 the left- and right-hand sides both transform the same under changes of coordinates (as given in Lemma 1). This relation emphasizes that the external forces $\boldsymbol{\tau}$ must be represented in the same coordinates as the left-hand side generalized coordinates. In other words, under changes of variable, the external force units must transform as well.

3 Conserved quantities

This section explores some of the theoretical properties properties of Lagrangian mechanics. We explore how symmetries in the Lagrangian mathematically give rise to conserved quantities in the physical system and demonstrate that both the conservation of momentum and the conservation of energy are actually mathematical consequences of these symmetries in the Lagrangian.

These results are special cases of a much more general principle described by Noether's theorem connecting any differentiable symmetry to associated conservation laws Susskind (2011). In other words, the fundamental conservation laws physicists discovered by experimentation are actually written into the underlying mathematics of Lagrange's principle of stationary action.

3.1 Conservation of momentum: a mathematical consequence of spacial symmetry

A continuous spacial symmetry is a spacial direction of perturbation $\delta \boldsymbol{q}$ under which the Lagrangian doesn't change to the first order

$$\mathcal{L}(\boldsymbol{q} + \delta \boldsymbol{q}, \dot{\boldsymbol{q}} + \delta \dot{\boldsymbol{q}}) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{23}$$

where the equality is understood to hold only to the first order. We can write this more succinctly using the notation $\frac{\delta \mathcal{L}}{\delta \boldsymbol{q}} = 0$, where we define $\frac{\delta \mathcal{L}}{\delta \boldsymbol{q}}$ is the directional derivative of the Lagrangian in the perturbation direction

$$\frac{\delta}{\delta \boldsymbol{q}} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right] = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \delta \dot{\boldsymbol{q}}. \tag{24}$$

If this directional derivative is zero in the direction $\delta \boldsymbol{q}$, then by definition, the Lagrangian doesn't change in that direction to the first order.

Often, in analogy to the interpretation of $\boldsymbol{f} = m\ddot{\boldsymbol{x}} = \frac{d}{dt} m\dot{\boldsymbol{x}}$ as force being the change in momentum of the system, we denote the terms of the Euler-Lagrange equation as

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}}_{\boldsymbol{f}} = \frac{d}{dt} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}}_{\boldsymbol{p}} \tag{25}$$

where $\boldsymbol{f}$ is known as the *generalized* force and $\boldsymbol{p}$ is known as the *generalized* momentum. This reinterpretation gives us the following two equations[10]

$$\boldsymbol{p}^T = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \qquad \dot{\boldsymbol{p}}^T = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \quad . \tag{26}$$

Saying that the first order perturbation is zero, therefore, says that

$$\frac{\delta}{\delta \boldsymbol{q}} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \underbrace{\dot{\boldsymbol{p}}^T \delta \boldsymbol{q} + \boldsymbol{p}^T \delta \dot{\boldsymbol{q}}}_{\frac{d}{dt}\left(\boldsymbol{p} \delta \boldsymbol{q} \right)} = 0, \tag{27}$$

which, in words, tells us that the component of generalized momentum $\boldsymbol{p}$ in the direction of perturbation doesn't change with time. In general, we write the perturbation as $\delta \boldsymbol{q} = \epsilon \boldsymbol{v}(\boldsymbol{q})$, which is some infinitesimal scaling ϵ of a vector valued function $\boldsymbol{v}(\boldsymbol{q})$. The above discussion tells us that if $\delta \boldsymbol{q} = \epsilon \boldsymbol{v}(\boldsymbol{q})$ is continuous (differentiable) symmetry of the Lagrangian, then the quantity $\boldsymbol{p}^T \boldsymbol{v}(\boldsymbol{q})$ is conserved:

$$\frac{1}{\epsilon} \frac{d}{dt} \Big(\boldsymbol{p} \delta \boldsymbol{q} \Big) = \frac{d}{dt} \underbrace{\Big(\boldsymbol{p}^T \boldsymbol{v}(\boldsymbol{q}) \Big)}_{\text{Conserved}} = 0. \tag{28}$$

[10] Under the vector calculus convention we're following, multi-dimensional gradients are calculated as rows $\frac{\partial f}{\partial \boldsymbol{x}} = \nabla_{\boldsymbol{x}} f(\boldsymbol{x})^T$, so that the rows of Jacobians $\boldsymbol{J}_\phi = \frac{\partial \phi}{\partial \boldsymbol{q}}$ are transposed gradients of ϕ's individual outputs. The momentum relations, therefore, include a transpose since we typically think of the momentum variables as column vectors.

Specifically, if we describe a mechanical system in Cartesian coordinates, this expression can be shown to be the conservation of linear momentum. If, on the other hand we describe the system in polar coordinates, this quantity can be shown to be the conservation of *angular* momentum. Generally, every spacial symmetry we can find, based on the above analysis, corresponds to some conserved generalized momentum.

To summarize, this analysis says that **the conservation of momentum is a fundamental mathematical consequence of spacial symmetries in the Lagrangian.**

3.2 Conservation of energy: a mathematical consequence of time symmetry

We can perform a similar analysis using time derivatives to analyze the mathematical consequences time symmetries. Suppose, to take the most general stance, that the Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ can vary as a function of time. Then the time derivative of the Lagrangian is

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \quad &= \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial \mathcal{L}}{\partial t} \quad = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial t} \\ &= \dot{\boldsymbol{p}}^T\dot{\boldsymbol{q}} + \boldsymbol{p}\ddot{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial t} \\ &= \frac{d}{dt}\left(\boldsymbol{p}^T\dot{\boldsymbol{q}}\right) + \frac{\partial \mathcal{L}}{\partial t},\end{aligned}$$

where we have used Equations 26 to write this expression in terms of momentum variables. Rearranging this expression, we get

$$\frac{d}{dt}\underbrace{\left(\boldsymbol{p}^T\dot{\boldsymbol{q}} - \mathcal{L}\right)}_{\mathcal{H}(\boldsymbol{q},\dot{\boldsymbol{q}},t)} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{29}$$

where $\mathcal{H}$ is known as the Hamiltonian.[11] This equation says that the time rate of change of the Hamiltonian is the negative time partial of the Lagrangian. In particular, if the Lagrangian itself isn't explicitly a function of time, then this time partial derivative is zero. **So if the Lagrangian function doesn't depend explicitly on time then this quantity, the Hamiltonian, is constant, unchanging, as the system evolves with time.**

What is the Hamiltonian? For mechanical systems, the Lagrangian is the difference between kinetic and potential energies $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\dot{\boldsymbol{q}}) - \mathcal{V}(\boldsymbol{q})$. This tells us, first, that the Lagrangian isn't explicitly a function of time, so the Hamiltonian will be conserved in this case. But additionally, the kinetic energy

[11] The Hamiltonian is the fundamental construct of Hamiltonian mechanics, a further reformulation of Classical Mechanics who's geometric (symplectic) structure simplifies the theoretical analysis of the system. This document only explore the Hamiltonian in connection with the conservation of energy, but Hamiltonian mechanics is a very deep subject that has been central to the development of quantum mechanics Susskind (2011),

for mechanical systems is always some quadratic form $\mathcal{T} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}\dot{\boldsymbol{q}}$, where $\boldsymbol{M}$ is a generalized mass matrix, so $\boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \frac{\partial \mathcal{T}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{M}\dot{\boldsymbol{q}}$. The Hamiltonian, then, reduces to

$$\begin{aligned}\mathcal{H}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \boldsymbol{p}^T\dot{\boldsymbol{q}} - \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} - (\mathcal{T} - \mathcal{V}) \\ &= \left(\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}\right) + \mathcal{V} \\ &= \underbrace{\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}}_{\mathcal{T}} + \mathcal{V} \quad = \mathcal{T} + \mathcal{V}.\end{aligned}$$

Thus, the Hamiltonian for mechanical systems whose kinetic energy takes the form $\mathcal{T} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$ is just the total energy of the system $\mathcal{H} = \mathcal{T} + \mathcal{V}$. And since the Lagrangian of mechanical systems doesn't explicitly depend on time, the above analysis shows that the total energy of the system remains constant as the system evolves. In other words, **the conservation of energy is a fundamental mathematical consequence of the time symmetry of the Lagrangian.**

4 Contact analysis

Above in Section 2.1.3 we saw that the Principle of Stationary Action holds equally well for constrained systems described in terms of a minimal set of generalized coordinates as it does for unconstrained systems. This allows us to use the mathematical structure of the generalized coordinate map ϕ to obviate the need for explicitly computing constraint forces. But in robotics, when the constraints are caused by contact with the environment, we really would like to know what that constraint force is so that we can properly control the interaction. This section introduces Gauss's Principle of Least Constraint which gives us a convenient calculational tool based on the Method of Lagrange Multipliers to explicitly calculate both the constrained dynamics and the resulting constraint (contact) forces.

4.1 Gauss's principle and the Method of Lagrange Multipliers

Section 2.1.3 discusses the roll of constraint forces and d'Alembert's principle dictating that the constraint forces operate only orthogonally to the constraint surface, but that statement alone isn't sufficient to fully resolve what those constraint forces are. Mathematically, we saw that if $\boldsymbol{x} = \phi(\boldsymbol{q})$ is the generalized-coordinate map, d'Alembert's principle enforced that the constraint force $\boldsymbol{f}_c$ lie in the Jacobian $\frac{\partial \phi}{\partial \boldsymbol{q}}$'s null space. That null space, though, may be very high dimensional. Which of these possible orthogonal forces is the one actually chosen by nature to impose the constraint?

Gauss's *Principle of Least Constraint* tells us that nature simply chooses the smallest of those forces that keeps the motion consistent with the constraints.

We can generically represent constraints on a system in generalized coordinates $\boldsymbol{q}$ as

$$\boldsymbol{A}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ddot{\boldsymbol{q}} = \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{30}$$

For instance, for holonomic constraints of the form $g(\boldsymbol{q}) = 0$ that describe the constraint as the zero set of some differentiable constraint function g, differentiating the constraint twice with respect to time gives

$$\begin{aligned} &\frac{d^2}{dt^2} g(\boldsymbol{q}) = \frac{d}{dt}\left(\boldsymbol{J}_g \dot{\boldsymbol{q}}\right) = \boldsymbol{J}_g \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_g \dot{\boldsymbol{q}} \\ \Rightarrow \quad &\boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b} \end{aligned}$$

with $\boldsymbol{A} = \boldsymbol{J}_g$ and $\boldsymbol{b} = -\dot{\boldsymbol{J}}_g \dot{\boldsymbol{q}}$. Additionally, any non-holonomic constraint of the form $g(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$ can similarly be brought into this canonical form by taking one time derivative. Thus, although the above discussion of Lagrangian mechanics was ultimately restricted to holonomic constraints, this Least Constraint formulation applies to a much more general set of constraints.[12]

Under general constraints of the form given by Equation 30, Gauss's principle of least constraint tells us specifically that the observed constrained acceleration of the system $\ddot{\boldsymbol{q}}^*$ is as similar as possible to the unconstrained acceleration $\ddot{\boldsymbol{q}}_u$ that would have occurred had the constraints not been there, where this notion of "similar" is measured with respect to the metric defined by the system's generalized mass matrix $\mathcal{M}$:

$$\begin{aligned} \ddot{\boldsymbol{q}}^* = &\underset{\ddot{\boldsymbol{q}}}{\operatorname{argmin}} \frac{1}{2} \|\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_u\|^2_{\mathcal{M}} \\ &\boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b}. \end{aligned} \tag{31}$$

The Method of Lagrange Multipliers tells us that we can solve this equation analytically by solving the following system of equations

$$\begin{aligned} &\mathcal{M}(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_u) - \boldsymbol{A}^T \boldsymbol{\lambda} = 0 \\ &\boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b}. \end{aligned} \tag{32}$$

Rather than solving these equations generically here (a straightforward linear algebra calculation (see Udwadia & Kalaba (1996))), we instead show what they mean in terms of the rigid body robotic systems in the next section.

4.2 The floating base equations of motion with contacts

The optimization problem defined by Equation 31 is described in generalized coordinates $\boldsymbol{q}$ because the coordinate system really doesn't matter. Most broadly,

[12] The trade-off is that under more general constraints, we loose some of the higher level theoretical properties that we derived for Lagrangian mechanics, such as the relationship between symmetries and conserved quantities.

we can say that the principle holds for $3n$-dimensional systems of n particles, where n as above may be huge on the order of 10^{23}. Then in terms of these Cartesian accelerations $\ddot{\boldsymbol{x}}$, some of the constraints on the system $\boldsymbol{A}_s\ddot{\boldsymbol{x}} = \boldsymbol{b}_s$ arise out of rigid body *structural* constraints, while others $\boldsymbol{A}_c\ddot{\boldsymbol{x}} = \boldsymbol{b}_c$ arise from *contact* with the environment. Then the principle under this decomposition tells us we can characterize the observed accelerations of the system $\ddot{\boldsymbol{x}}^*$ as

$$\begin{aligned} \ddot{\boldsymbol{x}}^* = &\underset{\ddot{\boldsymbol{x}}}{\operatorname{argmin}} \frac{1}{2}\|\ddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}_u\|^2_{\widetilde{\mathcal{M}}} \\ &\boldsymbol{A}_s\ddot{\boldsymbol{x}} = \boldsymbol{b}_s \\ &\boldsymbol{A}_c\ddot{\boldsymbol{x}} = \boldsymbol{b}_c, \end{aligned} \tag{33}$$

where now $\widetilde{\mathcal{M}}$ is a huge diagonal matrix consisting simply of each particle's mass.

4.2.1 Structural reductions of the optimization problem

Since this characterization of the system behavior is an optimization problem, we can solve it in two parts. As a first step, we can resolve the structural constraints $\boldsymbol{A}_s\ddot{\boldsymbol{x}} = \boldsymbol{b}_s$. If our rigid body system is fundamentally d-dimensional (the dimensionality of the generalized coordinate system $\boldsymbol{q}$ above), then resolving these structural equations will reduce the system down to just a d-dimensional optimization problem since the rest of the dimensions are constrained by these structural constraints. Since simple changes of coordinates don't change where the minimizer of the optimization problem lies in terms of the underlying accelerations of the physical particles, we can easily just perform a change of variable in this new d-dimensional problem to show that it matches the problem as represented in generalized coordinate given in Equation 31.

In other words, Equation 31 may be viewed as a partially resolved version of the full problem given in Equation 33. Importantly, any way of finding the dynamics of the system in generalized coordinates, by this argument, may be viewed as way of solving for the subspace represented by the structural constraints $\boldsymbol{A}_s\ddot{\boldsymbol{x}} = \boldsymbol{b}_s$. In practice, we typically use Lagrangian mechanics and the Principle of Stationary Action discussed in Section 2.1 to calculate the equations of motion in order to jump directly to the reduced problem in Equation 31.

4.2.2 Floating base contact dynamics

For many legged systems, we can model their dynamics using a floating base variant of the equations of motion. These unconstrained equations, which we can derived using Lagrange's Principle of Stationary Action, are given by

$$\mathcal{M}\ddot{\boldsymbol{q}}_u + \boldsymbol{h} = \boldsymbol{S}\boldsymbol{\tau}, \tag{34}$$

where $\boldsymbol{S}$ is a matrix defining how the set of external forces $\boldsymbol{\tau}$ are distributed among all dimensions of the system. For instance, if the full set of generalized

coordinates of the system consist both of actuated joint angles $\boldsymbol{q}_{\text{jnt}} \in \mathbb{R}^d$ and the unactuated Cartesian position and orientation of the floating base of the robot $\boldsymbol{q}_{\text{pos}} \in \mathbb{R}^6$, then the full set of generalized coordinates is[13] $\boldsymbol{q} = [\boldsymbol{q}_{\text{pos}}; \boldsymbol{q}_{\text{jnt}}]$ and the "actuation" matrix $\boldsymbol{S}$ is simply a zero-padded identity matrix of the form $\boldsymbol{S} = [\boldsymbol{0}; \boldsymbol{I}]$ so that the applied forces act only on the actuated dimensions $\boldsymbol{S\tau} = [\boldsymbol{0}; \boldsymbol{\tau}]$.

The unconstrained accelerations of this system are, therefore, $\ddot{\boldsymbol{q}}_u = \mathcal{M}^{-1}(\boldsymbol{S\tau} - \boldsymbol{h})$. Under Gauss's principle, substituting this expression into the first-order optimality conditions of Equations 32 gives

$$\begin{aligned} \mathcal{M}\ddot{\boldsymbol{q}} + \boldsymbol{h} &= \boldsymbol{S\tau} + \boldsymbol{A}^T\boldsymbol{\lambda} \\ \boldsymbol{A}\ddot{\boldsymbol{q}} &= \boldsymbol{b}. \end{aligned}$$

These equations are the contact-constrained equations of motion for floating-based rigid body robotic systems. Solving for the Lagrange multipliers $\boldsymbol{\lambda}$ give us explicitly the constraint forces as $\boldsymbol{f}_c = \boldsymbol{A}^T\boldsymbol{\lambda}$. When the contacts are at the robot's end-effectors, and those end-effectors are constrained to be stationary relative to the ground, the matrix $\boldsymbol{A}$ is just the Jacobian of the forward kinematic map $\boldsymbol{A} = \boldsymbol{J}_\phi$, and we can interpret $\boldsymbol{\lambda}$ as the Cartesian forces at those end-effectors. The quantity $\boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{J}_\phi^T\boldsymbol{\lambda}$ in this case then defines how those end-effector forces transform into generalized forces in the generalized coordinate system of the robot.

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[13] The notation $[\boldsymbol{x}; \boldsymbol{y}]$ denotes a new vector or matrix formed by stacking $\boldsymbol{x}$ on top of $\boldsymbol{y}$.